Asymptotic Accuracy of the Equilibrium-Diffusion Approximation

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HEDLA2012 Tallahassee FL, May 2, 2012

Overview

Background

- Non-relativistic equilibrium-diffusion is a fundamental limit of non-relativistic radiation-hydrodynamics.
- It is associated with optically-thick media, strong radiation-material coupling, and non-relativistic material speeds.
- An asymptotic expansion of the non-relativistic radiation-hydrodynamic (NRRH) solution has previously been shown to satisfy the non-relativistic equilibrium-diffusion (NRED) equations to leading-order (Lowrie, Morel, and Hittinger).
- In the current work, we have shown that this asymptotic expansion of the NRRH solution satisfies the NRED equations to first-order with an $O(\epsilon^2)$ error.

The Asymptotic Procedure

- Start with NRRH equations coupled to fully relativistic lab-frame transport equation and non-dimensionalize these equations.
- Identify non-dimensional parameters of interest and scale these by positive or negative powers of a small parameter ε:

$$\mathcal{U} \equiv \frac{v_\infty}{c} \sim \mathcal{O}(\varepsilon)$$

$$\mathcal{R} \equiv \frac{\rho_\infty u_\infty^2}{a_r T_\infty^4} \sim \mathcal{O}(1)$$

$$\mathcal{L} \equiv \frac{\ell_\infty}{\lambda_{t,\infty}} \sim \mathcal{O}(\varepsilon^{-1})$$

$$\mathcal{L}_s \equiv \frac{\lambda_{t,\infty}}{\lambda_{s,\infty}} \sim \mathcal{O}(\varepsilon).$$

The Asymptotic Procedure

- Equate terms multiplying each power of ε and manipulate the resulting equations to find closed equations for the unknowns in the power-series expansion.
- The first-order expansion to the NRRH solution, $U_0 + \varepsilon U_1$, satisfies the NRED equations, but the second-order expansion does not.

Why This is Important

- Radiation-hydrodynamics models and discretizations should preserve the asymptotic accuracy of the NRRH equations in the NRED limit.
- Discretizations and models that are not asymptotic preserving to leading-order are usually impractical for calculations that are strongly asymptotic due to the need for arbitrarily excessive mesh resolution.
- Discretizations and models that are partially asymptotic preserving will not be as accurate as they should be for calculations that are strongly asymptotic.

Why This is Important

- For instance, the Wilson flux limiter for radiative diffusion,

$$D = 1/(3\sigma_t + \|\vec{\nabla}\mathcal{E}\|/\mathcal{E}) \ ,$$

  is known to limit too strongly.
- This can be explained by the fact that it preserves the NRED limit to leading order but not to first order, i.e., the solution departs from diffusion with order ε instead of order ε^2 (Morel).
- In contrast, the Larsen limiter,

$$D = 1/\sqrt{(9\sigma_t^2 + [\|\vec{\nabla}\mathcal{E}\|/\mathcal{E}]^2)} \ ,$$

  preserves the NRED limit to first-order and is more accurate in general than the Wilson limiter (Morel).

Comoving-Frame Diffusion Model

- Start with comoving-frame energy and momentum equations.
- Assume $\mathcal{P}_{r,0} = \frac{1}{3}\mathcal{E}_{r,0}$.
- Obtain Fick's Law by setting certain terms to zero.

$$\partial_t \mathcal{E}_0 - \partial_i \left(\frac{c}{3\sigma_t} \partial_i \mathcal{E}_0 \right) + \partial_i \left(v_i \mathcal{E}_0 \right) + \frac{1}{3} \mathcal{E}_0 \partial_i v_i = c\sigma_a \left(aT^4 - \mathcal{E}_0 \right) \quad ,$$

$$\mathcal{F}_{0,i} = -\frac{c}{3\sigma_t} \partial_i \mathcal{E}_0 \quad .$$

Lab-Frame Diffusion Model

- Start with lab-frame energy and momentum equations.
- Assume $\mathcal{P}_r = \frac{1}{3}\mathcal{E}_r$.
- Set the time-derivative of the radiation flux to zero.

$$\partial_t \mathcal{E} - \partial_i \frac{c}{3\sigma_t} \partial_i \mathcal{E} + \partial_i \left\{ v_i \left[\frac{4}{3}\mathcal{E} + \frac{\sigma_a}{\sigma_t} \left(aT^4 - \mathcal{E}_0 \right) \right] \right\} =$$

$$c\sigma_a \left(aT^4 - \mathcal{E}_0 \right) - \frac{\sigma_t}{c} v_i \mathcal{F}_{0,i} \, .$$

$$\mathcal{F}_i = -\frac{c}{3\sigma_t} \partial_i \mathcal{E} + v_i \frac{4}{3} \mathcal{E} + v_i \frac{\sigma_a}{\sigma_t} \left(aT^4 - \mathcal{E}_0 \right) \quad .$$

Comparison of Model Accuracy

- The energy and momentum equations in the comoving- and lab-frames are equivalent, i.e., they transform into one another neglecting higher-order v_i/c terms.
- The diffusion approximations in the comoving- and lab-frames are fundamentally different, i.e., they do not transform into one another.
- It has been suggested that the lab-frame approximation should be fundamentally flawed because the radiation intensity *does not* become isotropic in the lab-frame.
- However, our asymptotic analyses show that both approximations, when coupled to the Euler equations, preserve the NRED limit to first order.
- This indicates that they are equally valid in the NRED limit.

Conclusions

- Knowledge of the asymptotic accuracy of the NRED approximation enable us to fully evaluate the accuracy of radiation-hydrodynamics models and discretizations in the NRED limit.
- Failure to preserve the accuracy of the NRED limit can lead to models and discretizations with significant flaws.